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1980 J. Phys. A: Math. Gen. 13 2001

(http://iopscience.iop.org/0305-4470/13/6/022)

J. Phys. A: Math. Gen. **13** (1980) 2001–2006. Printed in Great Britain

Excitation spectrum of the relativistic quantum plasma

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Received 8 May 1979, in final form 20 November 1979

Abstract. Using a covariant Wigner function approach developed elsewhere, the excitation modes of a relativistic quantum plasma are obtained. With this technique, the derivation is quite similar to the usual derivation (i.e. non-quantum) and hence can be applied to more involved cases.

In this paper the covariant Wigner function formalism developed elsewhere (Hakim 1978) is applied to a rederivation of the dispersion relations of a relativistic quantum plasma. These relations have already been derived earlier by Tsytovich (1961) who used a semi-phenomenological treatment (see also Biskamp (1967) and Chin (1977)), by Jancovici (1962) who used a many-body formalism and considered the zero temperature case only, by Melrose (1972) who discussed some general properties, by Hakim and Heyvaerts (1978) who started from a BBGKY hierarchy where spin effects were neglected and, finally, by Delsante and Frankel (1978) who used a dielectric constant approach. Essentially, relativistic quantum plasmas do occur in astrophysics (electrons in white dwarfs, magnetosphere of pulsars, radiative era of the primeval universe, etc) and hence deserve particular studies.

Here a relativistic quantum BBGKY hierarchy is given and then truncated to give a covariant quantum Vlasov–Hartree equation. Throughout this paper only an electron plasma embedded in a smooth positive neutralising background is considered.

The dynamical equations describing the system are (i) the Dirac equations

$$\begin{cases} [\gamma\,.\,(\mathrm{i}\partial-eA)-m]\psi=0 & (1a)\\ \bar{\psi}[\gamma\,.\,(\mathrm{i}\partial+eA)+m]=0 & (1b)\end{cases}$$

for the electron-positron field, and (ii) the Maxwell equations written as

$$\Box A^{\mu}=4\pi\tilde{J}^{\mu}, \tag{2}$$

where $\tilde{J}^{\mu}$ is the four-current operator

$$\tilde{J}^{\mu}\equiv e\bar{\psi}\gamma^{\mu}\psi, \tag{3}$$

to which a gauge condition (see below) must be added. In the above equations the notation is standard (see e.g. Schweber 1962): the metric used has the signature $+---$, the velocity of light and Planck's constant divided by 2π are equal to one, and $e=-|e|$.

0305-4470/80/062001+06\$01.50

The statistical description of the plasma is dealt with by using the following *one-particle* covariant Wigner function,

$$F(x, p) = \frac{1}{(2\pi)^4} \int d^4R \exp(-ip \,.\, R)\langle \bar{\psi}(x+R/2) \otimes \psi(x-R/2)\rangle, \tag{4}$$

and other quantities of interest such as $\langle \tilde{F}(x,p)A^\mu(x')\rangle$, $\langle \tilde{F}(x,p)\otimes\tilde{F}(x',p')\rangle$, $\langle A^\mu(x)\otimes A^{\mu'}(x')\rangle$, etc (in these last expressions $\tilde{F}$ has the *same* definition as F in equation (4) *except* that the average value $\langle\ldots\rangle \equiv \mathrm{Tr}(\rho\ldots)$ is not taken; $\tilde{F}$ is a Wigner operator and ρ is the density operator).

From F one can easily calculate the average value of one-particle operators (Hakim 1978) such as the four-current (3); for instance, one has (Hakim 1978)

$$J^\mu \equiv \langle \tilde{J}^\mu\rangle = \mathrm{Sp} \int d^4p\, \gamma^\mu F \tag{5}$$

where Sp is the brace over spin indices.

From Dirac's equations (1) and the definition (4) one can easily derive operator equations for $\tilde{F}$ that generate a BBGKY hierarchy (Hakim 1978) for the various statistical quantities of interest, $F \equiv \langle\tilde{F}\rangle$, $\langle\tilde{F}\otimes A\rangle$, $\langle\tilde{F}\otimes A\otimes A\rangle$ etc. Here, only the first equations of the hierarchy are needed. They read

$$[i\gamma\,.\,\partial + 2(\gamma\,.\,p - m)]F(x,p) = \frac{2e}{(2\pi)^4}\int d^4x'\,.\,d^4p' \exp[-ip'\,.\,(x-x')]\langle\gamma\,.\,\tilde{F}(x, p-p'/2)A(x')\rangle \tag{6a}$$

$$F(x,p)[\overleftarrow{i\gamma\,.\,\partial} - 2(\gamma\,.\,p-m)] = \frac{-2e}{(2\pi)^4}\int d^4x'\,.\,d^4p' \exp[-ip'\,.\,(x-x')]\langle A(x')\,.\,\tilde{F}(x,p+p'/2)\gamma\rangle \tag{6b}$$

which must be solved simultaneously. In order to close the hierarchy the Hartree–Vlasov ansatz is used (it is briefly discussed below and in Hakim (1978)):

$$\langle\tilde{F}(x,p)\otimes A(x')\rangle \sim F(x,p)\otimes\langle A(x')\rangle. \tag{7}$$

Doing so, we obtain a closed equation for F, quite similar to the usual Vlasov equation to which one must join the average value of equation (2), i.e.

$$\Box\langle A^\mu(x)\rangle = 4\pi e\, \mathrm{Sp}\int d^4p\,\gamma^\mu F(x,p), \tag{8}$$

where use has been made of equation (5). $\langle A^\mu\rangle$ is thus a classical field in *this* approximation and hence the Lorentz gauge condition

$$\partial_\mu\langle A^\mu(x)\rangle = 0 \tag{9}$$

can be used.

In order to obtain the dispersion relations for electromagnetic waves propagating through the plasma, the above equations have—as usual—to be *linearised* about an equilibrium state, i.e. about

$$\begin{cases} \langle A^\mu(x)\rangle_{\mathrm{eq}} \equiv 0 & (10) \\ F_{\mathrm{ep}}(p) = (\gamma\,.\,p+m)f_{\mathrm{eq}}(p)/4m, & (11)\end{cases}$$

where F_{eq} is derived in Hakim (1978) and where $f_{\mathrm{eq}}(p)$ has been obtained in Hakim and Heyvaerts (1978) as

$$f_{\mathrm{eq}}(p)=\frac{2m}{(2\pi)^3}\sum_{\pm}\frac{\theta(\pm p^0)\delta(p^2-m^2)}{\exp[\pm\beta(u\,.\,p-\epsilon_f)]+1} \tag{12}$$

(in equation (12) u is the average four-velocity of the plasma and ϵ_f is its chemical potential (its Fermi energy whenever its temperature $T\equiv\beta^{-1}/k_{\mathrm{B}}$ is zero, where k_{B} is Boltzmann's constant)). Notice that the linearisation procedure

$$\begin{cases} F(x,p)\sim F_{\mathrm{eq}}(p)+F^{(1)}(x,p) \\ \langle A^{\mu}(x)\rangle\sim A^{\mu(1)}(x) \end{cases} \tag{13}$$

is essentially equivalent to the *random phase approximation* as has been noted long ago (see e.g. Jancovici (1962)). Once they have been linearised and Fourier transformed equations (6) read respectively

$$[\gamma\,.\,(p-k/2)-m]\hat{F}^{(1)}(k,p)=e\hat{A}^{(1)}(k)\,.\,\gamma F_{\mathrm{eq}}(p+k/2) \tag{14a}$$

$$\hat{F}^{(1)}(k,p)[\gamma\,.\,(p+k/2)-m]=e\hat{A}^{(1)}(k)\,.\,F_{\mathrm{eq}}(p-k/2)\,.\,\gamma. \tag{14b}$$

Particular solutions of these last equations are easily found to be

$$\hat{F}^{(1)}_a(k,p)=\frac{e\hat{A}^{(1)}_{\mu}(k)}{4m}\left(\frac{[\gamma\,.\,(p-k/2)+m]\gamma^{\mu}[\gamma\,.\,(p+k/2)+m]}{(p-k/2)^2-m^2}\right)f_{\mathrm{eq}}(+), \tag{15a}$$

$$\hat{F}^{(1)}_b(k,p)=\frac{e\hat{A}^{(1)}_{\mu}(k)}{4m}\left(\frac{[\gamma\,.\,(p-k/2)+m]\gamma^{\mu}[\gamma\,.\,(p+k/2)+m]}{(p+k/2)^2-m^2}\right)f_{\mathrm{eq}}(-), \tag{15b}$$

where use has been made of equation (11) and of the notation

$$f_{\mathrm{eq}}(\pm)\equiv f_{\mathrm{eq}}(p\pm k/2) \tag{16}$$

and where the necessary iϵ-terms have been omitted. The most general solutions of equations (14) are, respectively, of the general form

$$\hat{F}^{(1)}=\hat{F}^{(1)}_a+[\gamma\,.\,(p-k/2)+m]G_1(-), \tag{17a}$$

$$\hat{F}^{(2)}=\hat{F}^{(1)}_b+G_2(+)[\gamma\,.\,(p+k/2)+m], \tag{17b}$$

where the last terms of the right-hand sides of these equations represent the *arbitrary* solutions of the homogeneous equations (14); in equations (17) G_1 and G_2 are arbitrary 4×4 matrices that are functions of p and are respectively on the mass shells

$$(p-k/2)^2=m^2 \text{ for } i=1 \qquad\text{and}\qquad (p+k/2)^2=m^2 \text{ for } i=2. \tag{18}$$

The reason why $G_1(-)$, for instance, contains a $\delta(-)$ factor can be seen by applying the operator $[\gamma\,.\,(p-k/2)-m]$ to equation (17a) from the left. The first term vanishes since $\hat{F}^{(1)}_a$ is a solution while the second vanishes only if $G_1(-)$ also contains a $\delta(-)$ factor.

From the necessary identity of equations (17a) and (17b), one concludes that

$$\begin{aligned}\hat{F}^{(1)}&=\hat{F}^{(1)}_a+\hat{F}^{(1)}_b\\ &=-\frac{e}{8m}\left(\frac{[\gamma\,.\,(p-k/2)+m]\gamma\,.\,\hat{A}^{(1)}(k)[\gamma\,.\,(p+k/2)+m]}{k\,.\,p}\right)\\ &\quad\times[f_{\mathrm{eq}}(p+k/2)-f_{\mathrm{eq}}(p-k/2)]\end{aligned} \tag{19}$$

where use has been made of the fact that

$$(p \pm k/2)^2 - m^2 = \pm 2k \cdot p \qquad \text{when } (p \mp k/2)^2 = m^2,$$

valid only when both equations (18) hold.

Defining the polarisation tensor $\Pi^{\lambda\mu}(k)$ by

$$4\pi \hat{J}^{\lambda}_{(1)}(k) \equiv \Pi^{\lambda\mu}(k) \hat{A}^{(1)}_{\mu}(k), \tag{20}$$

where $\hat{J}^{\lambda}_{(1)}(k)$ is the Fourier transform of the perturbed four-current (see equation (5)),

$$4\pi \hat{J}^{\lambda}_{(1)}(k) = 4\pi e \operatorname{Sp} \int d^4p \gamma^{\lambda} \hat{F}^{(1)}(k, p), \tag{21}$$

one obtains

$$\Pi^{\lambda\mu}(k) = -\omega_p^2 K^{\lambda\mu} - \Omega_p^2 g^{\lambda\mu} - k^2 \Delta^{\lambda\mu}(k)(\omega_p^2/4n_{eq}) I \tag{22}$$

where use has been made of the following relations (see e.g. Landau and Lifschitz (1959)):

$$\operatorname{Sp}(\gamma^{\lambda}\gamma^{\mu}) = 4g^{\lambda\mu} \tag{23a}$$

$$\operatorname{Sp}(\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}) = 0 \tag{23b}$$

$$\operatorname{Sp}(\gamma^{\lambda}\gamma^{\rho}\gamma^{\mu}\gamma^{\nu}) = 4(g^{\lambda\rho}g^{\mu\nu} - g^{\lambda\mu}g^{\rho\nu} + g^{\lambda\nu}g^{\rho\mu}). \tag{23c}$$

In equations (22) we have set† (as in Hakim and Heyvaerts (1978))

$$K^{\lambda\mu} = \frac{1}{n_{eq}} \int d^4p\, p^{\lambda} p^{\mu} \left(\frac{f_{eq}(+) - f_{eq}(-)}{k \cdot p + i\epsilon} \right) \tag{24}$$

$$I = \int d^4p \left(\frac{f_{eq}(+) - f_{eq}(-)}{k \cdot p + i\epsilon} \right) \tag{25}$$

$$\Omega_p^2 = \frac{4\pi e^2}{m} \int d^4p\, f_{eq}(p). \tag{26}$$

This last expression is nothing but the relativistic quantum form of the plasma frequency: as in the non-quantum but relativistic case (Hakim and Mangeney 1968, 1971) it has *not* its usual form $\omega_p^2 \equiv 4\pi n_{eq} e^2/m$, n_{eq} being the invariant equilibrium electron density. In equation (22) $\Delta^{\lambda\mu}(k)$ is the projector over the space orthogonal to k^{λ}:

$$\Delta^{\lambda\mu}(k) \equiv g^{\lambda\mu} - k^{\lambda}k^{\mu}/k^2. \tag{27}$$

Comparing now equation (22) with the expression calculated in Hakim and Heyvaerts (1978) (where spin effects were neglected; equation (4.20)), one can see that they differ by the last term only and a naive dimensional analysis of the latter shows that it is negligible when

$$k \ll p^* = \begin{cases} \text{Fermi momentum/energy (degenerate case)} \\ \text{Inverse Compton thermal wavelength (non-degenerate case).} \end{cases}$$

† In equations (24) and (25) the $i\epsilon$-factors of the resonant denominator $(k \cdot p)^{-1}$ have been re-established: they correspond to the usual Landau prescription; as a result these equations acquire an imaginary part that is vanishing as long as the waves are superluminous.

This is the reason why, in the zero temperature case, we found (Hakim and Heyvaerts 1978) anew Jancovici's (1962) results.

Using now the Lorentz gauge condition (9) under the form $k \cdot \hat{A}^{(1)} = 0$, and also equations (24)–(26), the dispersion equations are obtained from

$$[k^2 g^{\lambda\mu} - \Pi^{\lambda\mu}(k)]\hat{A}^{(1)}_\mu(k) = 0 \tag{28}$$

and found to be

$$-\frac{\Omega_p^2}{\omega^2 - \boldsymbol{k}^2} + \frac{\omega_p^2}{\omega^2 - \boldsymbol{k}^2} K^{11} - \frac{\omega_p^2}{4n_{\text{eq}}} I = 0 \tag{29}$$

for transverse modes and

$$1 - \frac{\Omega_p^2}{\omega^2 - \boldsymbol{k}^2} - \frac{\omega_p^2}{\omega^2 - \boldsymbol{k}^2} K^{00} + \frac{\omega}{|\boldsymbol{k}|} \frac{\omega_p^2}{\omega^2 - \boldsymbol{k}^2} K^{30} - \frac{\omega_p^2}{4n_{\text{eq}}} I = 0 \tag{30}$$

for longitudinal modes. These equations are, of course, identical to the ones previously derived. Note also that $k_\mu \Pi^{\mu\nu}(k) = 0$, as it should be.

At this point several remarks are in order.

First, equations (29) and (30) reduce to the classical relativistic equations (Hakim and Mangeney 1968, 1971) when $\hbar \to 0$: to see this, (i) neglect spin, i.e. I, (ii) suppress the contributions of the positrons and (iii) suppress the +1 of the Fermi factor in equation (12) (see Hakim and Heyvaerts (1978)); take the long wavelength limit.

A second remark deals with the absence of vacuum contributions either in equation (22) or in equations (29) and (30): indeed, in the absence of matter $f_{\text{eq}}(p)$ goes to zero. This is due to the fact that we have implicitly used a normal ordering of our field operators, thereby killing all vacuum contributions; furthermore this is also (Hakim 1978) due to our Hartree–Vlasov ansatz (7). Actually, if we don't omit the vacuum contribution to $f_{\text{eq}}(p)$ we have to replace $f_{\text{eq}}(p)$ (equation (12)) by its expression *plus* the *vacuum Wigner function*

$$f_{\text{vac}}(p) = \frac{2m}{(2\pi)^3}\,\theta(-p^0)\delta(p^2 - m^2). \tag{31}$$

Inserting this last expression into equation (22), for instance, gives rise to the usual vacuum polarisation tensor at order e^2. Notice that $F_{\text{vac}}(p)$ is given by an expression quite similar to that of $F_{\text{eq}}(p)$ (equation (11)); the calculation of $f_{\text{vac}}(p)$ is performed with $\rho_{\text{vac}} \equiv |\text{vac}\rangle\langle\text{vac}|$; equation (31) expresses the fact that the Dirac ocean of negative energy electrons (i.e. the vacuum) is uniformly filled. In the approximation used in this paper the renormalisation procedure merely reduces to a simple replacement of the bare charge and the bare mass by their (finite) experimental values: such a circumstance (except, of course, for charge renormalisation) also occurs in the classical (i.e. non-quantum but relativistic) context (Hakim 1967). The vacuum polarisation tensor has to be renormalised in the usual way (see e.g. Schweber (1962)).

Let us also notice that the above results could have been obtained by considering a quantised electron–positron field in the presence of an average self-consistent electromagnetic field: instead of using Dirac's equations (1) we would just have to use similar equations with A replaced by $\langle A\rangle$, acting as an external field. However such a procedure (like the one considered in this paper) does not take radiative corrections into account. It is known (Schwinger 1951a, b, c) that, in the presence of an *external* field $\langle A\rangle$, the wavefunction of an electron (positron) obeys a Dirac equation that takes into account its self electromagnetic field (radiative corrections) through a mass operator M:

$$[\gamma\,.\,(\mathrm{i}\partial - e\langle A\rangle) - M]\psi = 0 \tag{32a}$$

$$\bar{\psi}[\gamma\,.\,(\mathrm{i}\partial + e\langle A\rangle) + M] = 0 \tag{32b}$$

$$M\psi \equiv \int M(x, x')\psi(x')\,\mathrm{d}^4x'. \tag{32c}$$

For instance, at order e^2 $M(x, x')$ is given by (Schwinger 1951a, b, c)

$$M(x, x') = m\delta(x - x') + \mathrm{i}e^2\gamma G(x, x')\gamma D_+(x - x') \tag{33}$$

where $G(x, x')$ is the Green function of the electron in the presence of the external field $\langle A\rangle$ and D_+ is the photon propagator. Equations (32) lead to a modification (of order e^2) to the relativistic quantum Vlasov–Hartree equation. Such a modification can also be obtained from the relativistic quantum BBGKY hierarchy by an expansion in powers of e^2 (notice that, besides the usual plasma parameter, there also exist two other dimensionless parameters for a relativistic quantum plasma, i.e. mc^2/kT and $e^2/\hbar c$). These modifications are studied in a subsequent paper (Hakim 1980, unpublished).

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